On I-Ring Family$(\mathrm{IR}-\mathrm{F})$

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Abstract:

In this research we introduce at the first time (as we know), the concept of I-ring family. Also we introduce the concept of each of (sub family of I-ring family, , principalI-ring family and quotient family of I-ring family),H-connected,H-Hausdorffas well as a homomorphism , open and continuous morphism of I-ring families with some properties and examples about these concepts.

المستخلص :

في هذا البحث قدمنا و لأول مرة(حسب علمنا) مفهوم عائلة حلقة المثاليات كذلك قدمنا مفهوم كل من (العائلة الجزئية لعائلة الحلقة و عائلة الحلقة المتولدة بعنصرواحد و عائلة القسمة لعائلة الحلقة) بالإضافة الى مفهومالتشاكلات و الدوال المفتوحه المستمرة بين العوائل الحلقية .الاتصال نوع (H) كما تم التوصل إلى بعض النتائج المهمة والشيقة حول الموضوع.

Introduction:

Our aim in this research is to give a comprehensive study of I-ring family. On the other hand, we investigate about sufficient or necessary conditions for family to be I-ring family. The research consists of two sections.

In section one we reviewed some basic definitions and properties about ring, sub ring, ideals, principal and quotient ring, the definitions and properties about homomorphism of rings as well as some basic definitions and properties about topological concepts which we needed later in our work.

In section two we introduced the concept of a group family, where (A group family $(\mathrm{IR}-\mathrm{F})$ of a ring$(\mathrm{R},+,\cdot)$is the collection $(\mathrm{IR}-\mathrm{F}) = \{(\mathrm{I_i},+,\cdot):(\mathrm{I_i},+,\cdot)$ ideals of$(\mathrm{R},+,\cdot)$ $\}$satisfying that;

1) $(\mathrm{R},+,\cdot)$ and $(\{0\},+,\cdot) \in (\mathrm{IR}-\mathrm{F})$.
2) $\bigcap_{i=1}^{n}\big((\mathrm{I_i},+,\cdot)\big) \in (\mathrm{IR}-\mathrm{F})$ where $(\mathrm{I_i},+,\cdot) \in (\mathrm{IR}-\mathrm{F})$.
3)$\langle \bigcup_{i=1}^{n}(\mathrm{I_i},+,\cdot)\rangle \in (\mathrm{IR}-\mathrm{F})$ where $(\mathrm{I_i},+,\cdot) \in (\mathrm{IR}-\mathrm{F})$.

Also we introduce the definition of each of (sub family of group family, principal ring family, quotient family of ring family, as well as continuous , open and homomorphism of group family with some properties and examples about these concepts.

Section One: Prilimaries

This section contains some well-known definitions and remarks which are needed in our study. We don't remark the number of references with each definition in this section because these definitions in each reference remark in this research.

Definition (1.1) :[3] , [4]A ring is a triple$(\mathrm{R},+,\cdot)$ of a non empty set R and two binary operation$+$ and $\cdot$ satisfing that

1) $(\mathrm{R},+)$is a commutative group,
2) $(\mathrm{R},\cdot)$is a semi group ,
3)$(\cdot)$ distributive on $(+)$.

Definition (1.2): [3] , [4],[7]A sub ring$(\mathrm{S},+,\cdot)$of a ring$(\mathrm{R},+,\cdot)$ is a subset ofR which is itself ring .

Definition(1.3): [3] , [4],[5] A ring$(\mathrm{R},+,\cdot)$ is said to be commutative if, $x\cdot y = y\cdot x \ \forall\ x,y \in \mathrm{R}$.

Definition (1.4): [3] , [4] A center of a ring$(\mathrm{R},+,\cdot)$ is a subset $\mathrm{Z(R)} = \{x\in \mathrm{R}: x\cdot y = y\cdot x \ \ \forall y \in \mathrm{R}\}$ which isideal of a ring$(\mathrm{R},+,\cdot)$.

Definition (1.5): [3] , [4],[5]An ideal$(I,+,\cdot)$ is called principal if $\exists\, x\epsilon R$ such that $I=\langle x\rangle$.
Definition (1.6): [3] , [4] Let $(R,+,\cdot)$ and $(R',+',\cdot')$ be two rings. A mapping $f:(R,+,\cdot)\to(R',+',\cdot')$is called ahomomorphism if $f(x+y)=f(x)+f(y)$ and $f(x\cdot y)=f(x)\cdot f(y)\forall\ x,y\ \epsilon\, R$.
Proposition (1.7): [3] , [4]If $f(R,+,\cdot)\to(R',+',\cdot')$is a homomorphism then $(Ker(f),+,\cdot)$and $(Image(f),+,\cdot)$ are ideals of $(R,+,\cdot)$ and $(R',+',\cdot')$ respectively.
Definition (1.8): [3] , [4] A sub ring$(I,+,\cdot)$ of a ring$(R,+,\cdot)$ is said to be ideal of $(R,+,\cdot)$ if $x\cdot r$ and $x-y\ \epsilon\, I\ \forall\ x,y\ \epsilon\, I, r\,\epsilon\, R$.
Definition (1.9): [3] , [4],[5]Let $(I,+,\cdot)$ be an ideal of a ring$(R,+,\cdot)$ then the pair $((R/I),\otimes)$ form a ring called a quotient ring.
Definition (1.10): [1] , [2],[6]A topological space (X,τ) is connected if X and $\varnothing$ are the only sets in τ that are both open and closed.
Definition (1.11): [1] , [2],[6]A topological space (X,τ) is said to be Hausdorff space if any two points in X have nonintersecting neighborhoods.

Section Two: On Group Family:

The main concern of this section is to give a definition with some properties of ring families. We start with the following definition.

Definition (2.1):

An I-ring family$(IR-F)$ of a ring$(R,+,\cdot)$ is the collection $(IR-F)=\{(I_i,+,\cdot):(I_i,+,\cdot)$ ideals of$(R,+,\cdot)\}$satisfying that;
1) $(R,+,\cdot)$ and $(\{0\},+,\cdot)\ \in\ (IR-F)$.
2) $\bigcap_{i=1}^{n}(I_i,+,\cdot)\ \in(IR-F)$ where $(I_i,+,\cdot)\in(IR-F)$.
3)$\langle\bigcup_{i=1}^{n}(I_i,+,\cdot)\rangle\in(IR-F)$ where $(I_i,+,\cdot)\in(IR-F)$.

Examples and Remarks (2. 2):
1)The members of$(IR-F)$are called$(IR-F)$-open ideal and$(R,+,\cdot)$a mother of ideals in $(IR-F)$.
2) The pair $((R,+,\cdot)),(IR-F))$ is called I-ring family space.
3) For every ring $(R,+,\cdot)$ then $In(IR-F)=\{(R,+,\cdot),(\{0\},+,\cdot)\}$ is I-ring family known as indiscrete I-ring family. Each of $(R,+,\cdot)$ and$(\{0\},+,\cdot)$ open ideal in $In(IR-F)$.
4) For every ring$(R,+,\cdot)$ withall ideals of$(R,+,\cdot)$area ring family known as a discrete I-ring familyDi$(IR-F)$,all ideals of $(R,+,\cdot)$ open in Di$(IR-F)$.
5) For a ring $(Z_2,+_2,\cdot_2)$ then $(IR-F)=\{(Z_2,+_2,\cdot_2),(\{0\},+_2,\cdot_2)\}$ is a I-ring family
6) A family$(IR-F)=\{(Z,+,\cdot),(Z_e,+,\cdot),(\{0\},+,\cdot)\}$is I-ring family on $(Z,+,\cdot)$.
7) For a ring $(Z_p,+_p,\cdot_p)$ then $(IR-F)=\{(Z_p,+_p,\cdot_p),(\{0\},+_p,\cdot_p)\}$ is a I-ring family.p is prime .

Definition (2.3):

An ideal$(I,+,\cdot)$ of a ring$(R,+,\cdot)$ called $(IR-F)$closed ideal if there is $(I',+,\cdot)$open in $(IR-F)$such that $I\cup I'=G$ and$I\cap I'=\{0\}$.

Examples and Remarks (2.4):
1) In a discrete I-ring familyDi$(IR-F)$of a ring$(R,+,\cdot)$every ideal is closed.
2)In indiscrete I-ring family $In(IR-F)$ of a ring $(R,+,\cdot)$ there are two closed ideals $(R,+,\cdot)$ and$(\{0\},+,\cdot)$.
3) In I-ring family$(IR-F)=\{(Z,+,\cdot),(Z_e,+,\cdot),(\{0\},+,\cdot)\}$ the ring$(Z_e,+,\cdot)$is not closed.

Proposition (2.5):
The intersection of finite members of ring families $((IR-F))_i$ of a ring$(R,+,\cdot)$is I-ring familyof a ring$(R,+,\cdot)$
Proof: Let$(IR-F)=\bigcap_{i=1}^{n}((IR-F))_i$where$((IR-F))_i$I-ring familyes of a ring $(R,+,\cdot)$, then we must show that $(IR-F)$ is I-ring family of a ring$(R,+,\cdot)$. Now let $(I_i,+,\cdot)$ be open ideals in $((IR-F))_i$ by definition of I-ring family$((IR-F))_i$ then $(R,+,\cdot)$ and $\{0\},+,\cdot)\ \in\ ((IR-F))_i$

Therefore $(R,+,\cdot)$ and $\{0\},+,\cdot) \in (IR-F) = \bigcap_{i=1}^{n}(IR-F)_i$ also $(I_i,+,\cdot) \in ((IR-F))_i$ where $(I_i,+,\cdot) \in ((IR-F))_i$,and $\langle \bigcup_{i=1}^{n}(I_i,+,\cdot)\rangle \in (IR-F)$ where $(I_i,+,\cdot) \in (IR-F)$. also each of $\bigcap_{i=1}^{n}(I_i,+,\cdot) \in (IR-F) = \bigcap_{i=1}^{n}((IR-F))_i$ and $\langle \bigcup_{i=1}^{n}(I_i,+,\cdot)\rangle \in (IR-F) = \bigcap_{i=1}^{n}((IR-F))_i$ then $(IR-F) = \bigcap_{i=1}^{n}((IR-F))_i$ is a ring

Remarks (2.6):

The union of finite members of I-ring families $((IR-F))_i$ of a ring$(R,+,\cdot)$ is not I-ring family of a ring$(R,+,\cdot)$ in general.

Example (2. 7):

Let$(IR-F) = \{(Z_2,+_2,\cdot_2),(\{0\},+_2,\cdot_2)\}$. And let $(IR'-F')_Z = \{(Z,+,\cdot),(\{0\},+,\cdot)\}$ a ring Sub family of $(IR'-F')$
$(IR-F) \cup (IR'-F')_Z$is not I-ring family

Definition (2.8):

A ring sub family $((ISR-F))$ of a ring family $((IR-F))$ of a ring$(R,+,\cdot)$ is a sub family of $((IR-F))$ which is an I-ring family itself.

Examples (2. 9):

1) Each I-ring family is a ring Sub family of itself.
2) Let $((IR-F)) = \{(Z,+,\cdot),(Z_e,+,\cdot),(\{0\},+,\cdot)\}$ be a ring family on $(Z,+,\cdot)$,then $(ISR-F)_Z = \{(Z_e,+,\cdot),(\{0\},+,\cdot)\}$a ring Sub family of $(ISR-F)$

Proposition (2.10):

Let $((IR-F))$ be I-ring family of a ring$(R,+,\cdot)$, the triple $((IR-F),\cap,\langle\cup\rangle)$ is a ringon$((IR-F))$
1) $\cap$ is binary operation satisfy that $(I_1,+,\cdot),\cap\,(I_2,+,\cdot) \in ((IR-F))\forall\,(I_1,+,\cdot),(I_2,+,\cdot) \in ((IR-F))$,
2) (Associative law on $\cap$) $\forall\,(I_1,+,\cdot),(I_2,+,\cdot),(I_3,+,\cdot) \in ((IR-F))$, then $(I_1,+,\cdot) \cap ((I_2,+,\cdot) \cap (I_3,+,\cdot)) = ((I_1,+,\cdot) \cap (I_2,+,\cdot)) \cap (I_3,+,\cdot)$
3) There is an element $(R,+,\cdot) \in ((IR-F))$ such that$(I_1,+,\cdot) \cap (R,+,\cdot) = (R,+,\cdot) \cap (I_1,+,\cdot) = (I_1,+,\cdot)\ \forall\,(I_1,+,\cdot) \in ((IR-F))$,
4) For each element $(I_1,+,\cdot) \in ((IR-F))$ there is an element $\langle \bigcup_{i=1}^{n}(I_i,+,\cdot)\rangle \in ((IR-F))$ such that $\langle (I_1,+,\cdot) \cap \bigcup_{i=1}^{n}(I_i,+,\cdot)\rangle = (I_1,+,\cdot) \cap \langle \bigcup_{i=1}^{n}\big((I_i,+,\cdot)\big)\rangle = (R,+,\cdot) \in ((IR-F))$.
5) $(I_1,+,\cdot) \cap (I_2,+,\cdot) = (I_2,+,\cdot) \cap (I_1,+,\cdot)$ (Commutative law on $\cap$)
6) $\langle\cup\rangle$ is binary operation satisfy that $(I_1,+,\cdot)\langle\cup\rangle(I_2,+,\cdot) = \langle (I_1,+,\cdot) \cup (I_2,+,\cdot)\rangle \in ((IR-F))\forall\,(I_1,+,\cdot),(I_2,+,\cdot) \in (IR-F)$
7) $(I_1,+,\cdot)\langle\cup\rangle\big((I_2,+,\cdot)\langle\cup\rangle(I_3,+,\cdot)\big) =$
$((I_1,+,\cdot)\langle\cup\rangle(I_2,+,\cdot))\langle\cup\rangle(I_3,+,\cdot)\forall\,(I_1,+,\cdot),(I_2,+,\cdot),(I_3,+,\cdot) \in (IR-F)$,(Associative law on $\langle\cup\rangle$)
8)$(I_1,+,\cdot)\langle\cup\rangle\big((I_2,+,\cdot) \cap (I_3,+,\cdot)\big) = \big((I_1,+,\cdot)\langle\cup\rangle(I_2,+,\cdot)\big) \cap ((I_1,+,\cdot)\langle\cup\rangle(I_3,+,\cdot))$

Definition (2.11):

A commutative ring family $(Com(IR-F))$ is I-ring family $((IR-F))$ it is mother $(R,+,\cdot)$ is commutative.

Examples (2.12):

1) A family$(IR-F) = \{(Z,+,\cdot),(Z_e,+,\cdot),(\{0\},+,\cdot)\}$is a $(Com(IR-F))$ on$(Z,+,\cdot)$.
2) For a ring$(Z_p,+_p,\cdot_p)$ then $(IR-F) = \{(Z_p,+_p,\cdot_p),(\{0\},+_p,\cdot_p)\}$ is a$(Com(IR-F))$.p is prime .

Definition (2.13):

A principal I-ring family$(Pr(IR-F))$ is I-ring family of a ring$(R,+,\cdot)$ in which each idealis principal.

Examples (2.14):
1) A family$(\mathrm{IR}-\mathrm{F})_{\mathrm{Z}} = \{(\mathrm{Z},+,\cdot),(\{0\},+,\cdot)\}$is a principal I-ring family on$(\mathrm{Z},+,\cdot)$.
2) A family$(\mathrm{IR}-\mathrm{F}) = \{(\mathrm{Z},+,\cdot),(\mathrm{Z_e},+,\cdot),(\{0\},+,\cdot)\}$ is aprincipal I-ring family on$(\mathrm{Z},+,\cdot)$.

Remarks (2.15):
1) Every principal ideal I-ring family is commutative.
2) Every principal ring Sub family of a principal I-ring family is principal.

Definition (2.16):
An I-ring family$((\mathrm{IR}-\mathrm{F}))$of a ring $(\mathrm{R},+,\cdot)$ is called simple if $((\mathrm{IR}-\mathrm{F}))$ has no non-trivial ideal sub family.

Examples (2.17):
1) Each indiscrete I-ring family is a simple I-ring family.
2) For a ring $(\mathrm{Z_p},+_{\mathrm{p}},\cdot_{\mathrm{p}})$ then $(\mathrm{IR}-\mathrm{F}) = \{(\mathrm{Z_p},+_{\mathrm{p}},\cdot_{\mathrm{p}}),(\{0\},+_{\mathrm{p}},\cdot_{\mathrm{p}})\}$ is is a simpleI-ring family.

Definition (2.18):
An I-ring family $((\mathrm{IR}-\mathrm{F}))$ is called H-Hausdorff if for each pair of distinct points $\mathrm{a},\mathrm{b} \neq 0$ in$(\mathrm{R},+,\cdot)$ there is $((\mathrm{IR}-\mathrm{F}))$- open ideals $(\mathrm{I}',+,\cdot)$and$(\mathrm{I},+,\cdot)$ such that $(\mathrm{I},+,\cdot)\cap(\mathrm{I}',+,\cdot)=\{0\}$ and $\mathrm{a}\in(\mathrm{I},+,\cdot)$, $\mathrm{b} \in (\mathrm{I}',+,\cdot)$

Remark(2.19):
1) Each indiscrete I-ring familyIn$((\mathrm{IR}-\mathrm{F}))$ is not H-Hausdorff I-ring family.since for each pair of distinct points $\mathrm{a},\mathrm{b} \in (\mathrm{R},+,\cdot)$ there is no open ideal contain a and b either than $(\mathrm{R},+,\cdot)$ in $\mathrm{In}((\mathrm{IR}-\mathrm{F}))$
2) Each discrete I-ring family is H-Hausdorff ,since for each pair of distinct points $\mathrm{a},\mathrm{b} \in (\mathrm{R},+,\cdot)$ there is $(\mathrm{Di}(\mathrm{IR}-\mathrm{F}))$ - open ideals $(\mathrm{I}',+,\cdot)$ and $(\mathrm{I},+,\cdot)$ such that $(\mathrm{I},+,\cdot)\cap(\mathrm{I}',+,\cdot)=\{0\}$then $\mathrm{a}\in(\mathrm{I},+,\cdot)$ and $\mathrm{b} \in (\mathrm{I}',+,\cdot)$

Definition (2.20):
A quotient I-ring family $\mathrm{Q}(\mathrm{IR}-\mathrm{F})$ of a ring $(\mathrm{R},+,\cdot)$ is thecollection $(\mathrm{Q}(\mathrm{IR}-\mathrm{F})) = \{((\mathrm{R},+,\cdot)/(\mathrm{I},+,\cdot)),\otimes): (\mathrm{I},+,\cdot)\text{ideal of}(\mathrm{R},+,\cdot)\}$

Examples (2.21):
A quotient I-ring family$\mathrm{Q}(\mathrm{IR}-\mathrm{F})$ of a ring a family $(\mathrm{IR}-\mathrm{F}) = \{(\mathrm{Z},+,\cdot),(\mathrm{Z_e},+,\cdot),(\{0\},+,\cdot)\}$is $\mathrm{Q}(\mathrm{IR}-\mathrm{F}) = \{(\mathrm{Z},+,\cdot)/(\mathrm{Z_e},+,\cdot),(\mathrm{Z},+,\cdot)/(\mathrm{Z_e},+,\cdot),((\mathrm{Z},+,\cdot)/(\mathrm{Z_e},+,\cdot))\}$ I-ring familyon. $(\mathrm{IR}-\mathrm{F})$also
$\mathrm{Q}(\mathrm{IR}-\mathrm{F}) = \{(\mathrm{Z},+,\cdot)/(\{0\},+,\cdot),(\mathrm{Z},+,\cdot)/(\{0\},+,\cdot),((\mathrm{Z},+,\cdot)/(\{0\},+,\cdot)\}$ I-ring familyon$(\mathrm{IR}-\mathrm{F})$.

Definition (2.22):
A mapping $\mathrm{f}:(\mathrm{IR}-\mathrm{F}) \to (\mathrm{IR}'-\mathrm{F}')$ is continuous if for all $(\mathrm{I_i},+,\cdot)$ open in $(\mathrm{IR}'-\mathrm{F}')$ then$(\mathrm{f}^{-1}(\mathrm{I_i},+,\cdot))$ openin $((\mathrm{IR}-\mathrm{F}))$.
1)There is a continuous mapping from each I-ring family into itself.
2) A natural morphism is a continuous mapping from I-ring family$((\mathrm{IR}-\mathrm{F}))$intoA quotient I-ring family$\mathrm{Q}(\mathrm{IR}-\mathrm{F})$
3) A mapping$\mathrm{f}:\mathrm{Di}(\mathrm{IR}-\mathrm{F}) \to \mathrm{In}\,(\mathrm{IR}'-\mathrm{F}')$is an open mapping.

Definition (2.23):
A mapping$\mathrm{f}:(\mathrm{IR}-\mathrm{F}) \to (\mathrm{IR}'-\mathrm{F}')$is open if for all$(\mathrm{I_i},+,\cdot)$open in $(\mathrm{IR}-\mathrm{F})$ then$\mathrm{f}((\mathrm{I_i},+,\cdot)$ open in $(\mathrm{IR}'-\mathrm{F}')\,'$.
1)There is an open mapping from each I-ring family into itself.
2) A natural morphism is an open mapping from I-ring family$((\mathrm{IR}-\mathrm{F}))$intoA quotient I-ring family$\mathrm{Q}(\mathrm{IR}-\mathrm{F})$
3) A mapping $\mathrm{f}:\mathrm{In}(\mathrm{IR}-\mathrm{F}) \to \mathrm{Di}\,(\mathrm{IR}'-\mathrm{F}')$is an open mapping

Definition (2.24):

Let $(IR-F)$ and $(I'R'-F')$ be anI-rings of a ring $(R,+,\cdot)$ and $(R',+',\cdot')$ .respectivly amorphismf: $(IR-F) \rightarrow (I'R'-F')$is a homomorphism of a ring$(R,+,\cdot)$in to the ring $(R',+',\cdot')$.

1)There is amorphism from each I-ring family into itself.

2) A natural morphism is a morphism from I-ring family $((IR-F))$ intoA quotient I-ring family$Q(IR-F)$

Remark(2.25):

For each mapping of I-ring families the image and the converse image of identity and mother is identity and mother respectively.

Definition(2.26) :

An I-ring family$((IR-F))$ is H-connected if $(R,+,\cdot)$ and $(\{0\},+,\cdot)$ are the only open and closed ideals in $((IR-F))$.

Examples and Remarks (2. 27):

1) In a discrete I-ring family$Di(IR-F)$of a group $(R,+,\cdot)$ every ideal is closed then $Di(IR-F)$ is disconnected.

2) In indiscrete I-ring family$I(IR-F)$of a ring$(R,+,\cdot)$ there are two closed ideals $(R,+,\cdot)$and $(\{0\},+,\cdot)$then $In(IR-F)$ is connected.

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